Journal of Applied Mechanics and Technical Physics, Vol. 49, No. 5, pp. 846–852, 2008

SOLUTION OF CONTACT PROBLEMS ON THE BASIS OF A REFINED THEORY OF PLATES AND SHELLS

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UDC 539.3

Solutions of contact mixed boundary-value problems for a plate and for a cylindrical shell are given. These solutions are obtained with the use of equations for shells constructed by expanding solutions of elasticity theory equations with respect to the Legendre polynomials. Results of numerical simulations of the stress state in the vicinity of points with changing conditions on the frontal faces of the shell are presented. The results obtained are compared with analytical solutions of elasticity theory problems and with solutions obtained on the basis of the classical equations of the shell theory.

Key words: *layered shells, Legendre polynomials, contact problems, stress–strain state.*

Introduction. In contrast to shell theory equations derived on the basis of kinematic and force hypotheses [1], equations of shells constructed by means of approximating solutions of elasticity theory equations by segments of series of the Legendre polynomials [2–6] allow conditions on the frontal faces of the shell to be imposed not only in stresses but also in displacements. Therefore, these equations are used to solve mixed problems of the theory of plates and shells, which include contact problems. In the present paper, we solve contact problems with the use of refined equations of the theory of plates and shells; the algorithm used to construct them is described in [5, 6].

1. Equations of the Elastic Layer in the First Approximation. The approximation of stresses and strains by segments of the Legendre polynomials and the algorithm of constructing elastic layer equations are described in [5, 6]. In the present paper, we give only the final form of the elastic layer equations in the first approximation and the notation used in [5]. In constructing the equations, we use the notions of the basic and additional quantities (these quantities are coefficients at the Legendre polynomials in segments of series approximating stresses and strains). The differential equations of the elastic layer involve the derivatives of the basic quantities with respect to spatial variables. Vectors of normal and transverse forces, bending moments and torques, or corresponding displacements and rotation angles are defined on the end faces of the shell for equations in the first approximation.

The following approximations are used for displacements and stresses:

$$
\begin{gathered}
u(\xi,\zeta) = u_0(\xi) + u_1(\xi)P_1(\zeta) + u_2(\xi)P_2(\zeta) + u_3(\xi)P_3(\zeta),\\
v(\xi,\zeta) = v_0(\xi) + v_1(\xi)P_1(\zeta) + v_2(\xi)P_2(\zeta),\\
\sigma_1(\xi,\zeta) = t_1(\xi) + m_1(\xi)P_1(\zeta), \qquad \sigma_2(\xi,\zeta) = t_2(\xi) + m_2(\xi)P_1(\zeta),\\
\sigma_{12}(\xi,\zeta) = t_{12}(\xi) + m_{12}(\xi)P_1(\zeta) + r_{12}(\xi)P_2(\zeta).
\end{gathered}
\tag{1.1}
$$

Here ξ and ζ are the dimensionless longitudinal and transverse spatial coordinates in the layer, respectively, σ_1 and σ_2 are the dimensionless normal stresses in the transverse and longitudinal sections of the layer, respectively, and σ_{12} is the dimensionless shear stress.

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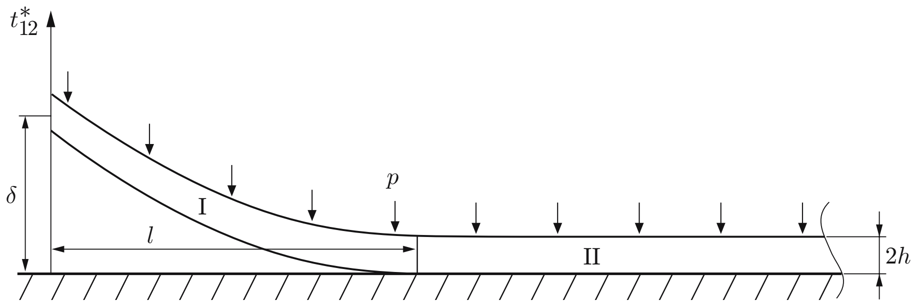

Fig. 1. Schematic of the problem of lifting of a semi-infinite band pressed to the base by uniform pressure: the regions without and with the band–base contact are marked as I and II, respectively.

The system of differential equations of the elastic layer with respect to the basic quantities in the first approximation is written as

$$\eta(t'_{11} + t_{13}) + (\sigma_{13}^+ - \sigma_{13}^-)/2 + q_1^0 = 0, \qquad \eta(t'_{13} + t_{11}) + (\sigma_{33}^+ - \sigma_{33}^-)/2 + q_3^0 = 0,$$

$$\eta m'_{11} - 3t_{13} + 3(\sigma_{13}^+ + \sigma_{13}^-)/2 + q_1^1 = 0,$$

$$t_{11} = \alpha(\eta(u'_0 + v_0) + \gamma v_1), \qquad t_{33} = \alpha(\gamma\eta(u'_0 + v_0) + v_1), \tag{1.2}$$

$$m_{11} = \alpha(\eta u'_1 + 3\gamma_1 v_2), \qquad m_{33} = \alpha(\gamma\eta u'_1 + 3v_2),$$

$$t_{13} = (\eta(v'_0 - u_0) + u_1 + u_3), \qquad m_{13} = 3(u_2 + v_1), \qquad r_{12} = 5mu_3,$$

where the prime denotes the derivative with respect to the variable ξ.

The system of 10 differential equations (1.2) with respect to 14 coefficients of approximations (1.1) is closed by four conditions on the layer surfaces at $\zeta = \pm 1$:

$$\begin{gathered}\sigma_2^\pm(\xi) = t_2(\xi) \pm m_2(\xi) \quad \text{or} \quad v^\pm(\xi) = v_0(\xi) \pm v_1(\xi) + v_2(\xi),\\ \sigma_{12}^\pm(\xi) = t_{12}(\xi) \pm m_{12}(\xi) + r_{12}(\xi) \quad \text{or} \quad u^\pm(\xi) = u_0(\xi) \pm u_1(\xi) + u_2(\xi) \pm u_3(\xi).\end{gathered} \tag{1.3}$$

In Eqs. (1.1)–(1.3), the stresses are normalized by the shear modulus μ; the following dimensionless quantities are introduced:

$$t_1 = \frac{1}{2h\mu}\int_{-1}^{1} \bar\sigma_1(\xi,\zeta)\,d\zeta, \qquad m_1 = \frac{3}{2h^2\mu}\int_{-1}^{1} \bar\sigma_1(\xi,\zeta)\zeta\,d\zeta, \qquad t_{12} = \frac{1}{2h\mu}\int_{-1}^{1} \bar\sigma_{12}(\xi,\zeta)\,d\zeta,$$

$$u = \frac{\bar u}{h}, \quad v = \frac{\bar v}{h}, \quad \xi = \frac{x}{l}, \quad \zeta = \frac{z}{h}, \quad \eta = \frac{h}{l}, \quad \alpha = \frac{2}{1-\gamma}, \quad \gamma = \frac{\nu}{1-\nu}.$$

Here the barred quantities are the dimensional stresses and displacements; x and z are the longitudinal and transverse spatial coordinates in the layer.

2. Problem of Separation of a Semi-Infinite Band Pressed to a Rigid Base by Uniform Pressure p. This problem refers to a class of contact problems with a decreasing contact zone (Fig. 1). The characteristic linear scale l in this problem is assumed to be the distance between the point of application of the lifting force and the contact point between the band and the base.

The following conditions are set on the frontal faces of the band at $\zeta = \pm 1$:

— in region I ($0 \le \xi \le 1$),

$$\sigma_2^+(\xi) = -p, \qquad \sigma_2^-(\xi) = 0, \qquad \sigma_{12}^\pm(\xi) = 0; \tag{2.1}$$

— in region II ($1 \le \xi < \infty$),

$$\sigma_2^+(\xi) = -p, \qquad v^-(\xi) = 0, \qquad \sigma_{12}^\pm(\xi) = 0. \tag{2.2}$$

The following conditions are set at $\xi = 0$:

$$t_1(0) = 0, \qquad m_1(0) = 0, \qquad t_{12}(0) = t^*_{12}. \tag{2.3}$$

As the shear stresses on the frontal faces of the band in this problem are equal to zero, the first equation of system (1.2) implies that $t_1(\xi) = \text{const}$.

The solution of system (1.2), (1.3) outside the contact region (region I in Fig. 1), which satisfies conditions (2.1)–(2.3), has the form

$$t_1 = 0, \quad t_{12} = t^*_{12} + \frac{p\xi}{2\eta}, \quad m_1 = \frac{3p\xi^2}{4\eta^2} + \frac{t^*_{12}\xi}{4\eta}, \quad t_2 = m_2 = -\frac{p}{2}, \quad m_{12} = 0, \quad r_{12} = -t_{12},$$

$$\begin{gathered} u_0 = u_0(0) + \frac{\gamma p\xi}{2\alpha\eta(1-\gamma^2)}, \qquad u_1 = u_1(0) + \frac{\gamma p\xi}{\alpha\eta(1-\gamma^2)} + \frac{p\xi^3}{4\alpha\eta^3(1-\gamma^2)}, \\ v_0 = v_0(0) - u_1(0)\frac{\xi}{\eta} + \Big(\frac{3p}{10\eta^2} - \frac{\gamma p}{4\alpha\eta^2(1-\gamma^2)}\Big)\xi^2 - \frac{p\,\xi^4}{16\alpha\eta^4(1-\gamma^2)}, \\ v_1 = -\frac{p}{2\alpha(1-\gamma^2)}, \qquad v_2 = -\frac{p}{6\alpha(1-\gamma^2)} - \frac{\gamma p\xi^2}{4\alpha\eta^2(1-\gamma^2)}. \end{gathered} \tag{2.4}$$

In the contact region (region II in Fig. 1), system (1.2), (1.3) can be reduced to a system of differential equations with respect to the functions $u_0(\xi)$, $u_1(\xi)$, and $v_0(\xi)$:

$$\begin{gathered} -3\gamma^2\eta^2 u_0''(\xi) + (4 - 3\gamma^2)\eta^2 u_1''(\xi) - 5(1-\gamma)u_1(\xi) - (5 - 2\gamma)\eta v_0'(\xi) \\ = -(3/2)\gamma(1-\gamma)\eta(\sigma_2^+)'(\xi) - 3\gamma\eta(v^-)'(\xi), \\ -9\gamma\eta u_0'(\xi) + (5 - 2\gamma)\eta u_1'(\xi) + 5(1-\gamma)\eta^2 v_0''(\xi) - 9v_0(\xi) \\ = -(9/2)\gamma(1-\gamma)\sigma_2^+(\xi) - 9v^-(\xi), \\ (4 - \gamma^2)\eta u_0'(\xi) - \gamma^2\eta u_1'(\xi) - 3\gamma v_0 = 2(1-\gamma)t_1(\xi) - (1/2)\gamma(1-\gamma)\sigma_2^+(\xi) + 3\gamma(v^-)(\xi). \end{gathered} \tag{2.5}$$

The characteristic equation of system (2.5) has the following form:

$$\lambda[(\lambda\eta)^4 - (3/10)(\gamma + 6)(\lambda\eta)^2 + 9/4] = 0. \tag{2.6}$$

The solution of system (2.5), which satisfies the condition of uniform compression of the band at infinity, is presented as

$$\begin{gathered} v_0 = \sum_{j=1}^{2} z_j l_j(\xi) - \frac{1}{2(1-\gamma)}(\gamma t_1 + p), \qquad u_1 = \sum_{j=1}^{2} z_j(U_3 l_j''' + U_1 l_j'), \\ u_0' = \sum_{j=1}^{2} z_j(U_2 l_j'' + U_0 l_j) + \frac{1}{\eta(1+\gamma)}(t_1 + \gamma p), \end{gathered} \tag{2.7}$$

where $l_1(\xi) = \exp(-A\xi/\eta)\cos(B\xi/\eta)$, $l_2(\xi) = \exp(-A\xi/\eta)\sin(B\xi/\eta)$, A and B are the real and imaginary parts of the roots of Eq. (2.6) $\lambda\eta = \pm A \pm iB$, and U_0, U_1, U_2, and U_3 are coefficients depending on the geometric parameter $\eta = h/l$ and on the parameter γ expressed via Poisson's ratio.

The remaining coefficients of expansions (1.1) of displacements and stresses with respect to the Legendre polynomials in region II are found from the following relations:

$$m_1 = \alpha\eta u_1' - 3\alpha\gamma(\gamma\eta(u_0' + u_1') - v_0)/4 + 3\gamma\sigma_2^+/4 + 3\alpha\gamma v^-/4,$$

$$m_{12} = 0, \qquad t_{12} = -r_{12} = 5(\eta v_0' + u_1)/6, \qquad t_2 = \alpha(\gamma\eta u_0' + v_1),$$

$$v_1 = (-\eta\gamma(u_0' + u_1') + 3v_0 - 3v^- + \sigma_2^+/\alpha)/4, \qquad v_2 = (-\eta\gamma(u_0' + u_1') - v_0 + v^- + \sigma_2^+/\alpha)/4,$$

$$u_2 = 0, \qquad u_3 = -(\eta v_0' + u_1)/6, \qquad \sigma_2^+ = 3\alpha(\gamma\eta u_0' + v_0 - v^-)/2 - (\alpha\gamma\eta u_1' - \sigma_2^-)/2.$$

TABLE 1

Value of $\lambda = \delta/(2p(1-\nu))$ as a Function of the Parameter η and Poisson's Ratio ν

η	Exact solution	First approximation		
		$\nu = 0.44$	$\nu = 0.33$	$\nu = 0.2$
0.05	162.80	167.44	166.99	166.61
0.10	26.20	27.62	27.48	27.36
0.15	9.90	10.65	10.56	10.51
0.20	5.27	5.75	5.70	5.65
0.25	3.37	3.72	3.68	3.64

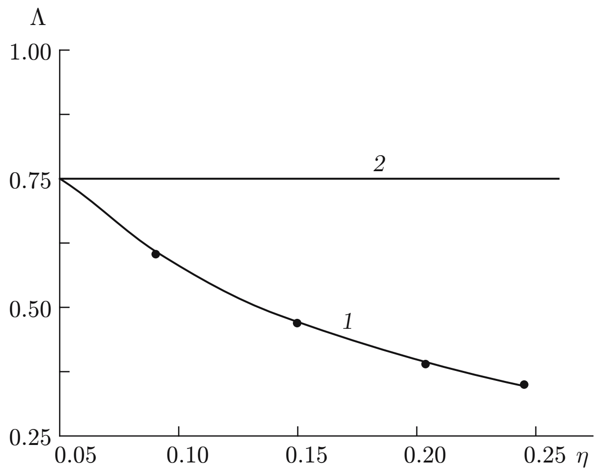

Fig. 2

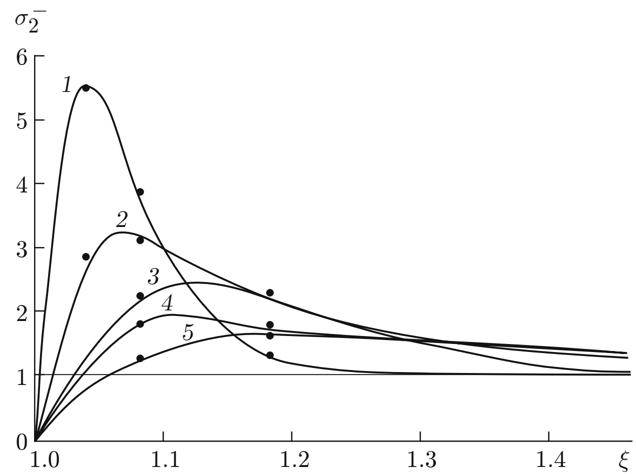

Fig. 3

Fig. 2. Dependence of Λ on the parameter η: curves 1 and 2 are the solutions of the layer equations in the first approximation and of the cylindrical bending theory equations; the points show the solution of the elasticity theory equations.

Fig. 3. Dimensionless contact pressure in the problem of lifting of a semi-infinite band for $\eta = 0.05$ (1), 0.1 (2), 0.15 (3), 0.2 (4), and 0.25 (5); the horizontal solid line is the solution of the equations of the cylindrical bending theory; the points show the solution of the elasticity theory equations.

Solutions (2.4), (2.7) contain the constants z_1, z_2, $u_0(0)$, $u_1(0)$, $v_0(0)$, and τ^*_{12}. The first five constants are determined from the condition of conjugation at the contact point $\xi = 1$:

$$m_1^1 = m_1^2, \qquad t_{12}^1 = t_{12}^2, \qquad u_0^1 = u_0^2, \qquad u_0^1 = u_0^2, \qquad v_0^1 = v_0^2.$$

The value of the lifting force τ^*_{12} applied at the point $\xi = 0$ is determined from additional conditions providing the uniqueness of the solution of equations in the contact problem:

$$v(\xi) \geq 0 \quad \text{at} \quad 0 \leq \xi \leq 1, \qquad \sigma_2^-(\xi) \geq 0 \quad \text{at} \quad \xi \geq 1.$$

Table 1 characterizes the value of $\lambda = \delta/(2p(1-\nu))$ as a function of the parameter $\eta = h/l$ for three values of Poisson's ratio ν. It should be noted that the value of λ in the exact solution (solution predicted by elasticity theory equations) is independent of the value of Poisson's ratio [7]. Such a dependence is observed in the solution of the layer equations in the first approximation, but the values of λ for different Poisson's ratios differ by less than 2% (see Table 1).

Figure 2 shows the value of $\Lambda = t^*_{12}(1-\nu)/(8\eta^3 v^-(0))$ as a function of the parameter η. The points show the solution of the elasticity theory equations, and the curve 2 is the solution of the equations of the cylindrical bending theory. Figure 3 shows the distribution of the dimensionless contact pressure for different values of the parameter η. The points show the solution of the elasticity theory equations [8].

It follows from Figs. 2 and 3 that the results obtained by solving the refined equations of shells and the elasticity theory equations are in reasonable agreement.

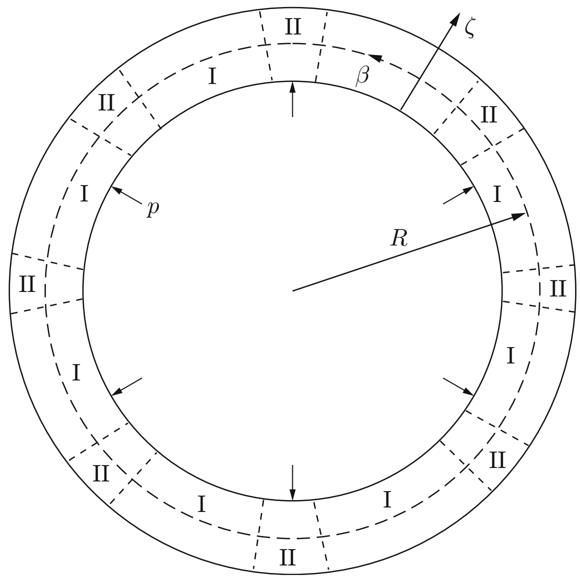

Fig. 4. Schematic of a contact problem for a cylindrical shell loaded by internal pressure: regions with different conditions on the frontal surfaces of the shell are marked as I and II.

3. Solution of a Mixed Boundary-Value Problem for a Cylindrical Shell. Results of the numerical solution of the problem of a stress–strain state of a cylindrical shell loaded by uniformly distributed internal pressure are given below (Fig. 4). A certain part of the shell (inner or outer) is assumed to be fixed.

The following dimensionless displacements and stresses are used:

$$u = \frac{U}{2h}, \qquad w = \frac{W}{2h}, \qquad \sigma_{11} = \frac{s_{11}}{\mu}, \qquad \sigma_{13} = \frac{s_{13}}{\mu}, \qquad \sigma_{33} = \frac{s_{33}}{\mu}.$$

Here U and V are the components of the displacement vector in directions 1 (β) and 3 (ζ), respectively; s_{11}, s_{13}, and s_{33} are the components of the stress tensor.

The following approximations over the shell thickness are used for dimensionless displacements and stresses:

$$\begin{gathered} u(\xi,\zeta) = u_0(\xi) + u_1(\xi)P_1(\zeta) + u_2(\xi)P_2(\zeta) + u_3(\xi)P_3(\zeta), \\ w(\xi,\zeta) = w_0(\xi) + w_1(\xi)P_1(\zeta) + w_2(\xi)P_2(\zeta), \\ \sigma_{11}(\xi,\zeta) = t_{11}(\xi) + m_{11}(\xi)P_1(\zeta), \\ \sigma_{13}(\xi,\zeta) = (t_{13} + \eta m_{13}(\xi)/3) + m_{13}(\xi)P_1(\zeta) + r_{13}(\xi)P_2(\zeta), \\ \sigma_{33}(\xi,\zeta) = t_{33}(\xi) + m_{33}(\xi)P_1(\zeta). \end{gathered} \tag{3.1}$$

The equations of equilibrium and relations of Hooke's law for the coefficients of expansions (3.1) yield

$$\begin{gathered} \eta(t'_{11} + t_{13}) + (m_{13} + \bar{q}_{10}) = 0, \qquad \eta(m'_{11} + m_{13}) + (3r_{13} + \bar{q}_{11}) = 0, \\ \eta(t'_{13} - t_{11}) + (m_{33} + \bar{q}_{30}) = 0, \\ t_{11} - a(\eta(u'_0 + w_0) + \gamma w_1 = 0, \qquad m_{11} - a(\eta u'_1 + 3\gamma w_2), \\ t_{22} - a\gamma(\eta(u'_0 + w_0) + w_1) = 0, \qquad m_{22} - a\gamma(\eta u'_1 + 3w_2) = 0, \\ t_{33} - a(\gamma\eta(u'_0 + w_0) + w_1) = 0, \qquad m_{33} - a(\gamma\eta u'_1 + 3w_2) = 0, \\ t_{13} - (\eta(w'_0 - u_0) + (u_1 + u_3)) = 0, \qquad m_{13} - 3u_2 - \eta u_3 = 0, \\ r_{13} - 5u_3 = 0. \end{gathered} \tag{3.2}$$

System (3.2) is closed by the boundary conditions on the inner and outer surfaces of the shell. As a result, the problem reduces to a boundary-value problem for a system of ordinary differential equations of the form

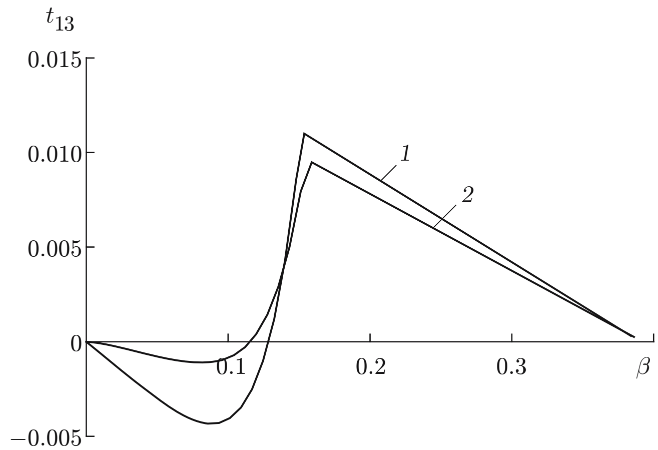

Fig. 5. Distribution of the shear force over the circumferential coordinate β: curves 1 and 2 show the results for conditions of zero displacements on the inner surface of the shell in regions of type II and zero displacements on the outer surface of the shell in regions of type II, respectively.

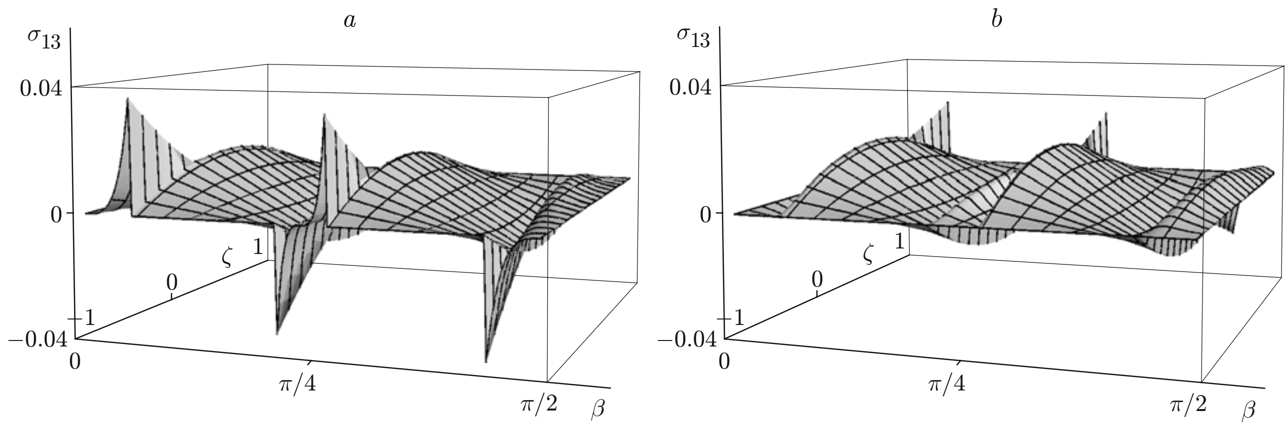

Fig. 6. Distributions of shear stresses over the shell cross section: (a) zero displacements on the inner surface of the shell in regions of type II; (b) zero displacements on the outer surface of the shell in regions of type II.

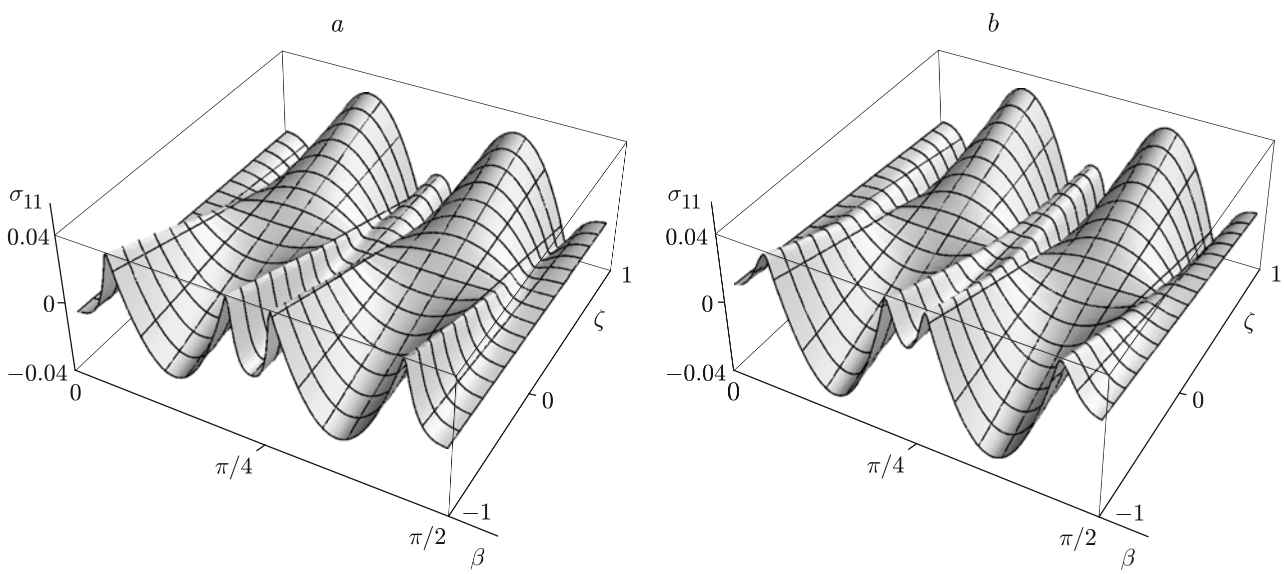

Fig. 7. Distributions of circumferential stresses over the shell cross section (notation the same as in Fig. 6).

$$Y' = AY + F, \tag{3.3}$$

where $Y = (u_0, u_1, w_0, t_{11}, m_{11}, t_{13})^{\rm t}$. The matrix A of the system and the vector of the right side F depend on the form of the boundary conditions on the shell surfaces. The boundary-value problem for system (3.3) was solved by the orthogonal sweep method.

In problems whose numerical solutions are demonstrated below, the conditions on the shell surfaces in regions I and II (see Fig. 4) are different. In regions I, uniformly distributed pressure p and zero shear stress were set on the inner surface, and zero normal and shear stresses were imposed on the outer surface. In regions II, zero displacements on the inner surface and zero stresses on the outer surface were set in the first problem. In the second problem, uniformly distributed pressure p and zero shear stress were imposed on the inner surface, and zero displacements were set on the outer surface. In both problems, there were eight regions of type II uniformly distributed over the circumferential coordinate β. The length of these regions was $\pi/10$. The boundary conditions for regions of type II modeled the supporting longitudinal stiffeners on the inner or outer surface of the shell (for instance, properly fixed stringers). The stresses arising in the vicinity of these stiffeners can lead either to plastic strains or to loss of continuity.

The problem was solved on the interval $0 \le \beta \le \pi/2$. Conditions of solution symmetry were set for $\beta = 0$ and $\beta = \pi/2$. The solutions of both problems are periodic with a period $\pi/8$. The interval $[0; \pi/2]$ was chosen to monitor the symmetry of the numerical solution.

Figure 5 shows the distributions of the shear force t_{13} over the circumferential coordinate β. The distributions of the shear stresses σ_{13} and circumferential stresses σ_{11} over the shell cross section are plotted in Figs. 6 and 7, respectively.

It follows from these results that the location of stiffeners (on the inner or outer surface) affects both the distributions of the shear and normal stresses and the intensity of stress concentration.

This work was supported by the Russian Foundation for Basic Research (Grant No. 08-01-00749) and by the Council on Grants of the President of the Russian Federation on the Support of Young Russian Scientists and Leading Scientific Schools (Grant No. NSh-3066.2008.1).

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